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# Characterizability and Uniqueness in Real Chebyshev Approximation

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In this note we study Chebyshev approximation by families of real continuous functions on a compact Hausdorff space $X$.

A sufficient condition (the closed-sign property) for best approximations to be characterized by the extrema of their error function is obtained. This condition is shown to be necessary if $X$ is perfectly normal. The closed-sign property is shown to be a sufficient condition for locally best approximations to be best, and to be a necessary condition if $X$ is perfectly normal. For families having the closed-sign property, a necessary and sufficient condition is obtained that best approximation always be unique. Less general non-uniqueness results are obtained for the case when approximations do not satisfy the characterization property.

## 1. INTRODUCTION

Let $X$ be a compact Hausdorff space and $C(X)$ be the space of real continuous functions on $X$ with norm

$$\|g\| = \text{Max}\,\{|g(x)| : x \in X\}.$$

Let $\mathscr{G}$ be a subset of $C(X)$, an *approximating family*, with elements $F$, $G$, $H$, $I$, .... The Chebyshev problem is: given $f \in C(X)$, find an element $G^* \in \mathscr{G}$ minimizing

$$e(G) = \|E(G, .)\|,$$

where $E(G,x) = f(x) - G(x)$, the error function. Such an element $G^*$ is called a best approximation in $\mathscr{G}$ to $f$ on $X$. Throughout the discussion, mention of $f$ is suppressed in the notations $e(G)$ and $E(G, .)$.

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## 2. Characterizability of Best Approximation

Let $M(G)$ be the set of points at which $|E(G,.)|$ attains its norm $e(G)$. As $X$ is compact and $E(G,.)$ is continuous, $M(G)$ is a non-empty compact set.

Lemma 1. *A sufficient condition for G to be a best approximation to f is that no $F \in \mathcal{G}$ exist such that*

$$E(G,x)(F(x)-G(x))>0 \qquad x \in M(G).$$

*Proof.* If $e(F)<e(G)$ then we have

$$\begin{aligned} f(x)-F(x) &< f(x)-G(x) && \text{if } f(x)-G(x)=e(G), \\ f(x)-F(x) &> f(x)-G(x) && \text{if } f(x)-G(x)=-e(G). \end{aligned}$$

If the condition of the lemma holds, no such $F$ can exist.

For most of the approximating families $\mathcal{G}$ of theoretical or practical interest it is known that the sufficient condition of the lemma is also necessary, which suggests the following

Definition. A family $\mathcal{G}$ of real continuous functions is *extremum characterizable* at $G \in \mathcal{G}$ if a necessary condition for $G$ to be best to any continuous function $f$ is that no $F \in \mathcal{G}$ exist such that $E(G,x)(F(x)-G(x))>0$ for all $x \in M(G)$. In case $\mathcal{G}$ is extremum characterizable at all of its elements, we say $\mathcal{G}$ is extremum characterizable.

We now define a property which, as we shall prove, implies extremum characterizability and which is equivalent to it for the common spaces $X$ of interest.

Definition. A family $\mathcal{G}$ of real continuous functions has the *closed-sign property* at $G \in \mathcal{G}$ if for any other element $F \in \mathcal{G}$ and any closed set $W$ on which $G-F$ has no zeros, there exists a sequence $\{H_k\} \subset \mathcal{G}$ converging uniformly to $G$ such that

$$\operatorname{sgn}(G(x)-H_k(x)) = \operatorname{sgn}(G(x)-F(x)) \qquad x \in W.$$

We shall say that $\mathcal{G}$ has the closed-sign property if $\mathcal{G}$ has it at all of its elements.

Families with the closed-sign property include families with the betweeness property (defined in [*2*]), which in turn include linear and rational families, and alternating families [*3*, especially 325–327]. As any dense subset of a family with the closed-sign property has itself this property, families with the closed-sign property need have no strong topological or interpolating properties. The closed-sign property is merely a very weak convexity property. Some examples of families without this property are given in Section 4.

If $\mathscr{G}$ does not have the closed-sign property at $G$, there exists an element $F \in \mathscr{G}$ and a closed set $W$ on which $F - G$ has no zeros such that

$$\mu = \inf\{\|G - H\| : H \in \mathscr{G}, \operatorname{sgn}(G(x) - H(x)) = \operatorname{sgn}(G(x) - F(x)), x \in W\}$$

is positive. We shall call such a triple $(F, W, \mu)$ a *closed-sign failure triple* for $G$.

THEOREM 1. *A sufficient condition for $\mathscr{G}$ to be extremum characterizable at $G$ is that $\mathscr{G}$ have the closed-sign property at $G$.*

*Proof.* Let $\mathscr{G}$ have the closed-sign property at $G$. Suppose that $F \in \mathscr{G}$ and that

$$E(G, x)(F(x) - G(x)) > 0 \qquad x \in M(G). \tag{1}$$

As $M(G)$ is closed, there exists by continuity of $E(G, .)(F - G)$ an open set $U''$ containing $M(G)$ such that (1) holds for $x \in U''$. As $X$ is a Hausdorff space, there exist open sets $U$ and $U'$ which have no point in common and which contain the closed sets $M(G)$ and $X \sim U''$, respectively. Now let $W$ be the closure of $U$. By the closed-sign property there exists an $H_k \in \mathscr{G}$ converging uniformly to $G$ such that

$$E(G, x)(H_k(x) - G(x)) > 0 \qquad x \in W.$$

Now let us choose $k$ so that $\|H_k - G\| < e(G)/2$; then it is readily seen that

$$|E(H_k, x)| < e(G) \qquad x \in U. \tag{2}$$

Let $V = X \sim U$. If $V$ is empty then $G$ is clearly nonbest, by (2). If $V$ is non-empty, it is a compact set containing no points of $M(G)$. It follows that

$$\mu = e(G) - \max\{|E(G, x)| : x \in V\}$$

is positive. Select $k$ so that $\|G - H_k\| < \max\{e(G)/2, \mu\}$. We have for $x \in V$,

$$|f(x) - H_k(x)| \leqslant |f(x) - G(x)| + |G(x) - H_k(x)| < e(G) - \mu + \mu = e(G),$$

and combining this with (2) we have

$$|E(H_k, x)| < e(G) \qquad x \in U \cup V = X.$$

As $E(H_k, .)$ is continuous we have $e(H_k) < e(G)$ and sufficiency is proven.

The sufficient condition turns out to be a necessary one for a class of compact spaces containing all the common compact spaces of interest, the perfectly normal compact spaces.

DEFINITION. A normal topological space in which each closed set is a countable intersection of open sets is called perfectly normal.

The perfectly normal spaces include all metric spaces.

THEOREM 2. *Let $X$ be a perfectly normal compact space. A necessary condition that $\mathcal{G}$ be extremum characterizable at $G$ is that $\mathcal{G}$ have the closed-sign property at $G$.*

*Proof.* Suppose $\mathcal{G}$ does not have the closed-sign property at $G$. Then there exists a closed-sign failure triple $(F, W, \mu)$ for $G$. Define $f$ by

$$f(x) = G(x) - \frac{\mu}{2}\operatorname{sgn}(G(x) - F(x))$$

for $x \in W$. By a corollary to Urysohn's lemma [*1*, p. 148], $f$ can be defined on $X \sim W$ so that it is continuous and $|f(x) - G(x)| < \mu/2$, $x \in X \sim W$. If an approximant $H$ such that $e(H) < e(G)$ did exist then it would satisfy the relations

$$\operatorname{sgn}(G(x) - H(x)) = \operatorname{sgn}(G(x) - F(x)) \qquad x \in W$$

and $\|G - H\| < \mu$. But this contradicts the definition of $\mu$ and so $G$ must be a best approximation. However, an approximant $F$ exists such that $E(G,x)(F(x) - G(x)) > 0$ for $x \in M(G) = W$. Hence failure of the closed-sign property at $G$ implies failure in being extremum characterizable at $G$. The theorem is proven.

In considering the Chebyshev minimization problem and in particular if we wish to apply descent methods, an important question is whether local minima are global ones.

DEFINITION. $G$ is a *locally best approximation* to $f$ if there exists a neighborhood $N$ of $G$ in $\mathcal{G}$ such that $e(G) \leqslant e(H)$ for all $H \in N$.

For the common approximating families, a locally best approximation is always a best approximation. We investigate to what extent this is true in general and find (as before) that the closed-sign property plays a key role.

DEFINITION. $\mathcal{G}$ is *globally minimizing* at $G$ if for any $f \in C(X)$, $G$ being a locally best approximation to $f$ implies that $G$ is a best approximation to $f$.

THEOREM 3. *Let $\mathcal{G}$ have the closed-sign property at $G$. Then $\mathcal{G}$ is globally minimizing at $G$.*

*Proof.* Suppose $e(F) < e(G)$. Then $E(G,x)(F(x) - G(x)) > 0$ for $x \in M(G)$. An examination of the proof of Theorem 1 shows that a sequence $\{H_k\} \subset \mathcal{G}$ exists, converging uniformly to $G$ and such that $e(H_k) < e(G)$. Hence $G$ is not locally best.

THEOREM 4. *Let $X$ be perfectly normal. If $\mathscr{G}$ is globally minimizing at $G$ then $\mathscr{G}$ has the closed-sign property at $G$.*

*Proof.* Let $\mathscr{G}$ not have the closed-sign property at $G$; then there exists a closed-sign failure triple $(F, W, \mu)$. As $F - G$ does not vanish on $W$, $W$ is a union of the two disjoint closed sets

$$W_1 = \{x : F(x) - G(x) > 0, x \in W\},$$
$$W_2 = \{x : F(x) - G(x) < 0, x \in W\}.$$

These are disjoint from the closed set $Z = \{x : F(x) = G(x)\}$. By the same corollary to Urysohn's lemma used before [*1*, p. 148], there exists a continuous function $g$ from $X$ into $[0,1]$ such that $g^{-1}(0) = Z$, $g^{-1}(1) = W_1$. Similarly there exists a continuous function $h$ from $X$ into $[0,1]$ such that $h^{-1}(0) = Z$, $h^{-1}(1) = W_2$. Define $f$ by

$$f(x) - G(x) \begin{cases} = g(x) \|F - G\| & \text{if } F(x) - G(x) > 0, \\ = -h(x) \|F - G\| & \text{if } F(x) - G(x) < 0, \\ = 0 & \text{if } F(x) = G(x). \end{cases}$$

Then $f$ is continuous and $e(G) = \|F - G\|$. As $f - G$, $F - G$ always have the same sign and cannot exceed $\|F - G\|$ in absolute value, $e(F) \leqslant \|F - G\|$. Equality can occur only if a point $x$ exists at which one of the pair $f(x) - G(x)$, $F(x) - G(x)$ vanishes and the other has absolute value $\|F - G\|$. No such point exists and so $e(F) < e(G)$. However $G$ is a locally best approximation, since a better approximation $H$ would satisfy

$$\operatorname{sgn}(G(x) - F(x)) = \operatorname{sgn}(G(x) - H(x)) \qquad x \in W.$$

As $(F, W, \mu)$ is a closed-sign failure triple for $G$, no such $H \in \mathscr{G}$ exists with $\|G - H\| < \mu$.

A consequence of Theorems 1–4 is

COROLLARY. *Let $X$ be perfectly normal. $\mathscr{G}$ is extremum characterizable at $G$ if and only if $\mathscr{G}$ is globally minimizing at $G$.*

We have thus connected the characterization problem and the minimization problem.

## 3. UNIQUENESS OF BEST APPROXIMATIONS IN FAMILIES WITH THE CLOSED-SIGN PROPERTY

DEFINITION. A pair $(G, H)$ of distinct elements of $\mathscr{G}$ is called *zero-sign compatible* if for any closed subset $Z$ of the zeros of $G - H$ and any $s \in C(X)$ taking the values $+1$ or $-1$ on $Z$, $\|s\| \leqslant 1$, there exists an $F \in \mathscr{G}$ such that

$\operatorname{sgn}(F(x)-G(x))=s(x)$ for $x \in Z$. If all pairs of distinct elements of $\mathscr{G}$ are zero-sign compatible we say that $\mathscr{G}$ has *zero-sign compatibility.*

LEMMA 2. *If a pair $(G, H)$ of distinct elements of $\mathscr{G}$ is not zero sign compatible then there exists a continuous function which has both G and H as best approximations.*

*Proof.* Let $Z$ be a closed subset of the zeros of $G-H$ and $s$ an element of $C(X)$ for which the zero-sign compatibility of $(G, H)$ fails. Define:

$$f(x)=G(x)+s(x)[\|G-H\|-|G(x)-H(x)|];$$

then

$$E(G, x)=s(x)[\|G-H\|-|G(x)-H(x)|].$$

For $x \in Z$ we have $E(G, x)=s(x)\|G-H\|$; hence $Z \subset M(G)$. If a better approximant $F$ existed it would satisfy

$$\operatorname{sgn}(F(x)-G(x))=s(x) \qquad x \in Z,$$

which is impossible by hypothesis. Hence $G$ is a best approximation to $f$ and since

$$\begin{aligned}|f(x)-H(x)| &\leqslant |f(x)-G(x)|+|G(x)-H(x)| \\ &\leqslant \|G-H\|-|G(x)-H(x)|+|G(x)-H(x)|=\|G-H\|,\end{aligned}$$

$H$ is also a best approximation to $f$.

LEMMA 3. *Let $\mathscr{G}$ have the closed-sign property at F and at G ($\neq F$), and let the pair $(F, G)$ be zero-sign compatible. Then there exists no continuous function for which both F and G are best approximations.*

*Proof.* Let us suppose that both $F$ and $G$ are best approximations to a continuous function $f$. Let $N$ be the set of points $x$ of $M(F)$ at which $F(x)=G(x)$. If $N$ were empty we would have

$$E(F, x)(G(x)-F(x))>0, \qquad x \in M(F)$$

which by extremum characterizability of $\mathscr{G}$ at $F$ implies that $F$ is not best. We now suppose that $N$ is non-empty. By continuity of $F-G$, $N$ is closed. By zero-sign compatibility of the pair $(F, G)$ there exists $I \in \mathscr{G}$ such that

$$(I(x)-G(x)) \cdot E(F, x)>0 \qquad x \in N.$$

By continuity of $(I-G) E(F, .)$ there exists a subset $U'$ of $M(F)$, open with respect to $M(F)$, on which the above inequality holds. There exists a subset $U$ of $U'$, open with respect to $M(F)$, such that $N \subset U \subset U'$ and

$\bar{U} \cap (M(F) \sim U') = \varnothing$. By the closed sign property at $G$ there exists a sequence $\{H_k\} \subset \mathscr{G}$ converging uniformly to $G$ such that

$$(H_k(x) - G(x))\, E(F, x) > 0 \qquad x \in \bar{U}.$$

As

$$(G(x) - F(x))\, E(F, x) \geqslant 0 \qquad x \in M(F), \tag{3}$$

we have

$$(H_k(x) - F(x))\, E(F, x) > 0 \qquad x \in \bar{U}.$$

Further, $V = M(F) \sim U$ is closed (and hence compact) and contains no points of $N$. By (3) we have

$$(G(x) - F(x)) \cdot E(F, x) > 0 \qquad x \in V.$$

By uniform convergence of $\{H_k\}$ to $G$ and by compactness of $V$, we have for all $k$ sufficiently large

$$(H_k(x) - F(x))\, E(F, x) > 0 \qquad x \in V.$$

Combining this with the previous inequality for $x \in \bar{U}$, we have for all $k$ sufficiently large

$$(H_k(x) - F(x))\, E(F, x) > 0 \qquad x \in U \cup V = M(F).$$

But as $\mathscr{G}$ is extremum characterizable at $F$ and as $F$ is best, we have a contradiction, proving the lemma.

From the two preceding lemmas we obtain

THEOREM 5. *Let $\mathscr{G}$ have the closed-sign property. A necessary and sufficient condition that best approximations to all continuous functions be unique is that $\mathscr{G}$ have zero-sign compatibility.*

## 4. UNIQUENESS WITHOUT THE CLOSED-SIGN PROPERTY

In the case $\mathscr{G}$ does not have the closed-sign property, the uniqueness problem is more difficult, particularly since there is no general theory for such $\mathscr{G}$. In the case that $\mathscr{G}$ has the closed-sign property at most of its elements we may be able to use the previous techniques to find conditions under which uniqueness of best approximations occurs. It is much easier, however, to find conditions under which non-uniqueness must occur. Lemma 2 is still applicable and enables us to detect many of the obvious cases in which non-uniqueness occurs for some function $f$. However $\mathscr{G}$ can have zero-sign compatibility without best approximations being unique.

EXAMPLE. Let $X$ be a two-point set. Then $C(X)$ can be represented by the set of all 2-tuples. Let $\mathcal{G} = \{(g_1, g_2) : g_1 g_2 = 0\}$. The zero element of $\mathcal{G}$ is the only element with the closed-sign property. As distinct approximants can agree on at most one of the two points, $\mathcal{G}$ has zero-sign compatibility. It is easily seen that the only functions having a unique best approximation are elements of $\mathcal{G}$.

An examination of the proof of Theorem 2 suggests a second way of finding non-uniqueness in case $\mathcal{G}$ does not have the closed-sign property.

LEMMA 4. *Let $(F, W, \mu)$ be a closed-sign failure triple for $G$. Let there exist an element $H \in \mathcal{G}$ such that $\|G - H\| = \mu$ and*

$$\operatorname{sgn}(G(x) - H(x)) = \operatorname{sgn}(G(x) - F(x)) \qquad or \qquad G(x) = H(x) \qquad x \in W.$$

*Then there exists a continuous function having $G$ and $H$ as best approximations.*

*Proof.* Note that by definition of $H$ we have

$$G(x) - \mu \leqslant H(x) \leqslant G(x) + \mu \tag{4}$$

In the proof of Theorem 2 it was shown that any continuous function $f$ satisfying

$$f(x) = G(x) - \frac{\mu}{2} \operatorname{sgn}(G(x) - F(x)) \qquad x \in W \tag{5}$$

$$\|f - G\| = \mu/2$$

has $G$ as a best approximation. In case we have $\|f - H\| \leqslant \mu/2$, $H$ is also a best approximation. We now show that such a continuous function $f$ exists.

By Urysohn's lemma a continuous function $f$ exists satisfying (5). Define a continuous function $g$ as follows:

$$g(x) \begin{cases} = H(x) + (\mu/2) & \text{if } |f(x) - H(x)| \leqslant \mu/2, \\ = f(x) & \text{if } f(x) - H(x) > \mu/2, \\ = H(x) - (\mu/2) & \text{if } f(x) - H(x) < -\mu/2. \end{cases}$$

By construction, $\|g - H\| \leqslant \mu/2$. For $x$ such that $g(x) = f(x)$, we have

$$|g(x) - G(x)| = |f(x) - G(x)| \leqslant \mu/2.$$

Next consider $x$ such that $g(x) = H(x) + (\mu/2)$. We have by the left inequality of (4)

$$g(x) - G(x) = H(x) + (\mu/2) - G(x) \geqslant -(\mu/2)$$
$$g(x) - G(x) = H(x) + (\mu/2) - G(x) < f(x) - G(x) \leqslant \mu/2,$$

giving $|g(x) - G(x)| \leqslant \mu/2$. Similarly for $x$ such that $g(x) = H(x) - (\mu/2)$ we also get $|g(x) - G(x)| \leqslant \mu/2$. We therefore have $\|g - G\| \leqslant \mu/2$. For $x \in W$, $H(x)$ lies in the closed interval between $G(x)$ and $G(x) - \mu \operatorname{sgn}(G(x) - F(x))$, and $f(x)$ is the midpoint of the interval; hence $|f(x) - H(x)| \leqslant \mu/2$ and by definition of $g$, $g(x) = f(x)$. So

$$g(x) = G(x) - \frac{\mu}{2} \operatorname{sgn}(G(x) - F(x)) \qquad x \in W$$

and $\|g - G\| = \mu/2$. Thus, a continuous function whose existence was asserted has been constructed and the lemma is proven.

Lemma 4 is a powerful and general result. It may however be difficult to guarantee the existence of $H$ without some compactness hypothesis. A suitable hypothesis is that closed bounded subsets of $\mathscr{G}$ are compact. This hypothesis is satisfied if $\mathscr{G}$ is any closed subset of a finite dimensional linear family or any closed subset of an $n$-parameter (unisolvent) family on an interval.

THEOREM 6. *Let $\mathscr{G}'$ be a family of continuous functions for which any closed bounded subset is compact. Let $\mathscr{G}$ be a subset of $\mathscr{G}'$. A necessary and sufficient condition that a unique best approximation exists to every continuous function is that $\mathscr{G}$ be closed, have the closed-sign property, and be zero-sign compatible.*

*Proof.* It follows by standard existence arguments that $\mathscr{G}$ being a closed subset of $\mathscr{G}'$ implies the existence of best approximations. By Theorem 5, the closed-sign property and zero-sign compatibility imply uniqueness. Sufficiency is proven. If $\mathscr{G}$ is not closed, let $f$ be an element of $\bar{\mathscr{G}} \sim \mathscr{G}$. Then $f$ has no best approximation. To assure existence of best approximations it is therefore, necessary that $\mathscr{G}$ be closed. Let $\mathscr{G}$ be closed but not have the closed-sign property. Then there exists a closed-sign failure triple $(F, W, \mu)$ for some element $G \in \mathscr{G}$. The set

$$S = \{H: \operatorname{sgn}(G(x) - H(x)) = \operatorname{sgn}(G(x) - F(x)), x \in W, \|G - H\| < 2\mu, H \in \mathscr{G}\}$$

is a non-empty bounded subset of $\mathscr{G}$. Its closure $\bar{S}$ is a non-empty closed bounded subset of $\mathscr{G}$ and of $\mathscr{G}'$, hence compact. Let $\{H_k\}$ be a sequence of elements of $S$ such that $\{\|G - H_k\|\}$ is decreasing, with limit $\mu$. Then $\{H_k\}$ has a subsequence converging to an element $H \in \bar{S}$ and $\|G - H\| = \mu$,

$$\operatorname{sgn}(G(x) - H(x)) = \operatorname{sgn}(G(x) - F(x)) \quad \textit{or} \quad G(x) = H(x) \qquad x \in W.$$

By Lemma 4 there exists a continuous function with both $G$ and $H$ as best approximations. Necessity is thus proven.

The combined existence-uniqueness problem for $\mathscr{G}$ a subset of a finite dimensional linear space, which includes all cases of approximation on a finite point set and which was raised by Rice in [*4*, p. 90–91], has been solved by the above theorem. It would be desirable to have a more explicit solution.

In case $\mathcal{G}$ is a nonlinear family, a unique best approximation can exist to all continuous functions without $\mathcal{G}$ having the closed-sign property.

EXAMPLE. Let $x = [0, 1]$ and $\mathcal{G} = \{F(a, .) : a \geqslant 0\}$,

$$F(0, x) = 0$$
$$F(a, x) = [1 + a]/(1 + x/a) \qquad a > 0.$$

$\mathcal{G}$ does not have the closed-sign property at the zero approximant. For $a < b$ we have $F(a, .) < F(b, .)$, hence

$$-e(a) \leqslant E(a, .) < E((a + b)/2, .) < E(b, .) \leqslant e(b),$$

and $e((a + b)/2) < \max\{e(a), e(b)\}$. It follows that there cannot exist two best approximations by elements of $\mathcal{G}$.

From the fact that any bounded sequence of elements of $\mathcal{G}$ has a subsequence converging pointwise to an element of $\mathcal{G}$ except possibly at the point 0, it follows by standard arguments that a best approximation exists to any continuous function.

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